Numerical Ranges and their Convexity

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Chapter 1

Introduction

1.1 Abstract

The study of numerical ranges plays a pivotal role in the analysis of linear operators on Hilbert spaces, with significant implications in functional analysis and quantum mechanics. Numerical ranges, often referred to as field-of-values, provide a geometric framework for understanding the behavior of operators. This paper explores various classes of numerical ranges, including the classical numerical range, the generalized numerical range. We examine their geometric properties, particularly focusing on their convexity. A primary result is that, while the classical numerical range is always convex, the convexity of other numerical ranges may vary depending on the operator and the underlying space. The paper provides insights into the conditions under which different numerical ranges exhibit convexity.

1.2 Notation

1. $\mathbb{F}$ denotes a scalar field
2. $\mathbb{R}$ denotes the field of Real Numbers
3. $\mathbb{C}$ denotes the field of Complex Numbers
4. $\mathbb{M}_n(\mathbb{F})$ denotes the set of all nxn matrices with entries in $\mathbb{F}$.(They also represent operators on F^n).
5. I_n represents the nxn identity matrix and O_n represent the nxn zero matrix.
6. $\mathbb{U}(n) = \{U \in M_n(\mathbb{C}) : U^*U = UU^* = I_n\}$ denotes the set of all unitary matrices.
7. $\mathbb{H}_n(\mathbb{F}) = \{H \in \mathbb{M}_n(\mathbb{F}) : H = H^*\}$ denotes the set of all Hermitian matrices. (It is a Vector space over $\mathbb{R}$)

Chapter 2

Numerical Range

2.1 Classical Numerical Range of an Operator

Let A$\in$M$_n$($\mathbb{C}$). Then the Numerical Range of A is described as the set W(A), where

$$W(A) := \{< x, Ax >: x \in \mathbb{C}^n \ and \ ||x|| = 1\}$$

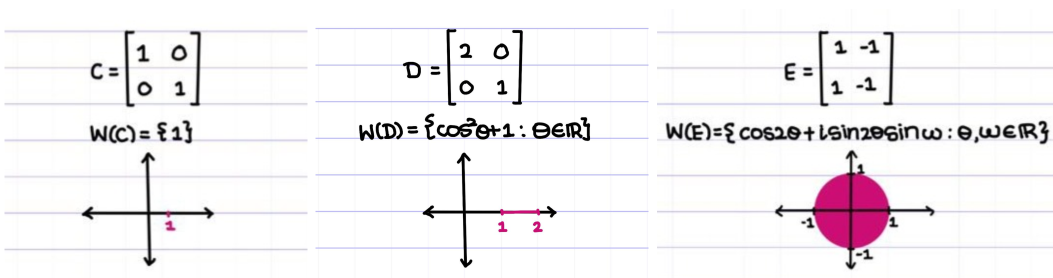

Figure 2.1: E.g of numerical ranges of a few operators.

2.2 Toeplitz-Hausdorff Theorem

For all A$\in$M$_n\mathbb{C}$, W(A) is convex .

Proof:

Let A$\in$M$_n\mathbb{C}$, x $\in \mathbb{C}^n$. Now, $< x, Ax >= x^*Ax = tr(Axx^*)$. If n=1, then if $||x|| = 1 \implies x = e^{i\theta}$ for some real θ and thus, W(A) = $\{tr(Ae^{i\theta}e^{-i\theta})\} = \{tr(A)\}$ which is convex.

For n$\geq$ 2, WLOG, it is enough to prove for n=2. Suppose it is true for n=2, then let m>2 and A$\in$M$_m\mathbb{C}$, x,y $\in \mathbb{C}^m$ such that $||x|| = ||y|| = 1$. Now in order to prove that W(A) is convex, it is enough to show that for some $\alpha \in (0, 1)$ there exists z $\in \mathbb{C}^m$ such that $||z|| = 1$ and $\alpha < x, Ax > +(1 - \alpha) < y, Ay >=< z, Az >$. Consider the projection map P: $\mathbb{C}^m \to$ span{x,y}. Then, if B= PAP$|_{Span\{x,y\}}$:$Span\{x, y\} \to Span\{x, y\}$, we get that [B] is an operator on $\mathbb{C}^2$. Thus, W(B) is convex. That is,for every $\alpha \in (0, 1)$ there exists z$\in$ $span\{x, y\}$ such that $||z|| = 1$ and $\alpha < x, Bx > +(1 - \alpha) < y, By >=< z, Bz >$. Now, for any w $\in$ $span\{x, y\}$ $< w, PAP(w) >=< w, PA(Pw) >=$

$< w, P(Aw) >=< w, Aw >$.

Now, for n=2, let $H_2 = \{M \in M_2(\mathbb{C}) : M = M^*\}$ be the space of all Hermitian Operators over $\mathbb{R}$. Consider the linear map $\Phi : H_2 \to \mathbb{C}$ given by $\Phi(M) = tr(AM)$. Now, it is enough to show that Φ maps the set of all 1-D ortho-projectors xx* onto a convex set.

$$x^*x = 1 \implies x = (cos\theta, e^{i\o} sin\theta)$$

$$\begin{aligned} xx^* &= \begin{bmatrix} cos^2\theta & cos\theta sin\theta e^{i\o} \\ cos\theta sin\theta e^{-i\o} & sin^2\theta \end{bmatrix} \\ &= (1/2) \begin{bmatrix} 1 + cos2\theta & sin2\theta e^{i\o} \\ sin2\theta e^{-i\o} & 1 - cos2\theta \end{bmatrix} \\ &= (1/2)I_4 + (1/2) \begin{bmatrix} cos2\theta & sin2\theta e^{i\o} \\ sin2\theta e^{-i\o} & -cos2\theta \end{bmatrix} \\ &= (1/2)I_4 + (1/2)M(\theta, \o) \end{aligned}$$

where $\{M(\theta, \o) : \theta, \o \in \mathbb{R}\}$ is a 2-Sphere in $\mathbb{R}^4$ centered at the origin.Now,the linear projection of a 2-Sphere into $\mathbb{C}$ can be either elliptical with it's interior, a line segment or a point, which, in any case, is convex. Therefore,since $\Phi(xx*) = (1/2)(tr(A) + (1/2)tr(AM(\theta, \o)))$, we get that W(A) is convex.

■

Chapter 3

Generalized Numerical Ranges

3.1 k-Numerical Range of an Operator

Let A∈$M_n(\mathbb{C})$. Then the k-Numerical range of A is described as the set $W_k(A)$, where

$$W_k(A) := \{\Sigma_{i=1}^{k} < x_i, Ax_i >: x_i \in \mathbb{C}^n \ and\ x_i s\ are\ orthonormal\}$$

3.1.1 Halmos-Berger theorem

For all A∈$M_n\mathbb{C}$, $W_k(A)$ is convex .

Proof:

Let A∈$M_n\mathbb{C}$, $1 \leq k \leq n$, $x_1, ..., x_k, y_1, ..., y_k \in \mathbb{C}^n$ such that $x_i's$ and $y_i's$ are both orthonormal, $\alpha \in (0,1)$.

Let M = span$\{x_1, ..., x_k\}$ and N = span$\{y_1, ..., y_k\}$. Let P:$\mathbb{C}^n \to M$ and Q:$\mathbb{C}^n \to N$ be two projection maps onto the respective spaces. Let T:M→N s.t. T(w)=QP(w).
Then by SVD, there exists U=$[u_1|...|u_k]$, V=$[v_1|...|v_k] \in U(k)$ (Note that U $\in L(M)$ and V $\in L(N)$) such that V*TU = D = diag$\{d_1, ..., d_k\}$ where $d_1 \geq ... \geq d_k \geq 0$.
Let L_i = span$\{u_i, v_i\}$. Claim: $L_i \perp L_j\ \forall i \neq j$.

$$\begin{aligned} < u_i, v_j > &=< Pu_i, Qv_j > \\ &=< QPu_i, v_j > \\ &=< d_i v_i, v_j > \\ &= d_i < v_i, v_j > \\ &= 0 \end{aligned}$$

whenever i≠j. Thus, $L_i \perp L_j\ \forall i \neq j$. Now, from Toeplitz-Hausdorff theorem, we have that for all $1 \leq i \leq n$, there exists $w_i \in L_i$ such that $||w_i|| = 1$ and $< w_i, Aw_i >= \alpha < u_i, Au_i > +(1-\alpha) < v_i, Av_i >$. Thus, $\sum_{i=1}^{k} < w_i, Aw_i >= \alpha \sum_{i=1}^{k} < u_i, Au_i > +(1-\alpha) \sum_{i=1}^{k} < v_i, Av_i >= \alpha \sum_{i=1}^{k} < x_i, Ax_i > +(1-\alpha) \sum_{i=1}^{k} < y_i, Ay_i >$. (Since M = $span\{x_1, ..., x_k\} = span\{u_1, ..., u_k\}$ and N = $span\{y_1, ..., y_k\} = span\{v_1, ..., v_k\}$). Hence, proved.

3.2 c-Numerical Range of an Operator

Let A$\in$M$_n$($\mathbb{C}$) and c=(c$_1$,...,c$_n$)^T $\in \mathbb{C}^n$.Then the c-Numerical Range of A is described as the set W$_c$(A), where

$$W_c(A) := \{\sum_{i=1}^{n} ci < x_i, Ax_i >: x_i \in \mathbb{C}^n \ and \ x_i s \ form \ an \ orthonormal \ basis\}$$

3.2.1 Result on convexity of c-Numerical Range

For all A$\in$M$_n$($\mathbb{C}$) and c $\in \mathbb{R}^n$, W$_c$(A) is convex .

(Proof in the next section)

3.3 C-Numerical Range of an Operator

Let A,C$\in$M$_n$($\mathbb{C}$).Then the C-Numerical Range of A is described to be the set W$_C$(A), where W$_C$(A)={tr(CUAU*): U$\in$ $\mathbb{U}(n)$}.

3.3.1 Properties of C-Numerical Range

For all A,C,U$\in$M$_n\mathbb{C}$, such that U is unitary, the following are true:

1. W$_C$(A)=W$_A$(C)
2. W$_{UCU^*}$(A)=W$_C$(A)
3. W$_C$(UAU*)=W$_C$(A)

Proof: (Direct from the definition)

3.3.2 Result on convexity of C-Numerical Range

For all A,C$\in$M$_n\mathbb{C}$, such that C is hermitian, W$_C$(A) is convex.

Proof:

Before we prove this, we shall prove the following lemmas.
Let A,C$\in$ $M_n(\mathbb{C})$ and U$\in$ $\mathbb{U}(n)$.

Lemma 1: If for some m: $1 \leq m < n$, C leaves invariant all m-dimensional subspaces of C^n, then C is a scalar.
Proof: Let x$\in$ $\mathbb{C}^n \backslash \{0\}$. Let $y_1, ..., y_{n-1}$ be an orthogonal basis of $\{x\}^{\perp}$. Let M = span$\{x, y_1, y_2, ..., y_{m-1}\}$ and N = span$\{y_1, y_2, ..., y_m\}$. Thus, $C(M) \subset M$ and $C(N) \subset N$. Therefore, $C(M \cap N) \subset M \cap N$. Now, $M \cap N$ = span $\{y_1, ..., y_{m-1}\}$. Thus, M = span{x}$\oplus(M \cap N) \implies$ C(span{x})$\subset$ span{x}. Thus, C leaves invariant all 1-dimensional subspaces of $\mathbb{C}^n$.
Let $\{b_1, ..., b_n\}$ be the standard basis of $\mathbb{C}^n$.Then, there exists $\alpha, \beta_1, ..., \beta_n \in \mathbb{C}$ such that $C(\sum_{i=1}^n b_i) = \alpha(\sum_{i=1}^n b_i)$ and $C(b_i) = \beta_i b_i$ for all i. Now, $\alpha(\sum_{i=1}^n b_i) = C(\sum_{i=1}^n b_i) = \sum_{i=1}^n C(b_i) = \sum_{i=1}^n \beta_i b_i$. Hence, we get that $\alpha = \beta_i$ for all i, which makes C = αI_n.

■

Lemma 2: If $CU^*AU = U^*AUC$ for all $U \in \mathbb{U}(n)$, then either A or C is a scalar.
Proof: Suppose A is not a scalar. Let λ be an eigenvalue of A. If E_λ is the corresponding eigenspace, then, since A is not a scalar, (m=) $\dim(E_\lambda) < n$. Let M be any subspace of $\mathbb{C}^n$ such that dim(M)=m. Then, there exists an unitary matrix U$\in \mathbb{U}(n)$ such that M = U*E$_\lambda$, which is the eigenspace of U^*AU corresponding to λ. Then, for any x $\in M$, $U^*AU(Cx) = C(U^*AUx) = C(\lambda x) = \lambda C(x)$. Thus, C(x)$\in M$ $\forall x \in M$, which means that C leaves invariant all m-dimensional subspaces of $\mathbb{C}^n$. Hence, by lemma 1, C is a scalar.

■

Lemma 3: If W$_C$(A) = $\{\alpha\}$, then C commutes with U^*AU for all U $\in \mathbb{U}(n)$, and hence, either C or A is a scalar.
Proof: Let S be skew-hermitian be fixed. Then for all $\theta \in \mathbb{R}$, $e^{\theta S}$ is unitary. Therefore, $W_C(A) = \{\alpha\} \implies tr(CU^*AU) = \alpha$ for all $U \in \mathbb{U}(n)$. Let V$\in \mathbb{U}(n)$. Therefore, $tr(Ce^{-\theta S}U^*AUe^{\theta S}) = \alpha$.

$$\begin{aligned}
\frac{d}{d\theta}(tr(Ce^{-\theta S}U^*AUe^{\theta S})) &= tr(\frac{d}{d\theta}(Ce^{-\theta S}U^*AUe^{\theta S})) \\
&= tr(-CSe^{-\theta S}U^*AUe^{\theta S} + Ce^{-\theta S}U^*AUe^{\theta S}S) \\
&= tr(Ce^{-\theta S}U^*AUe^{\theta S}S - e^{-\theta S}U^*AUe^{\theta S}CS) \\
&= tr((Ce^{-\theta S}U^*AUe^{\theta S} - e^{-\theta S}U^*AUe^{\theta S}C)S)
\end{aligned}$$

At $\theta = 0$, we get $tr((CU^*AU - U^*AUC)S) = 0$ for all skew-Hermitian S, U $\in \mathbb{U}(n)$. Now, every matrix B can be written as the linear combination of two skew-Hermitian Matrices: $B = \frac{1}{2}(B - B^*) - \frac{i}{2}[\frac{i}{2}(B + B^*)]$. Thus, $tr((CU^*AU - U^*AUC)B) = 0$ for every matrix B. Taking $B = PP_iP^T$ where P is any permutation matrix and P_i is the matrix whose entries are 1 at (i,i) and 0 elsewhere and varying them, we get that $CU^*AU = U^*AUC$ for all U $\in \mathbb{U}(n)$.

■

Lemma 4: If A$\in \mathbb{M}_2(\mathbb{C})$ and $\alpha = (\alpha_1, \alpha_2)$, then $W_\alpha(A)$ is convex.
Proof: Let x $\in C^2$ such that $||x|| = 1$. Then $\exists! y \in C^2 : y \in \{x\}^\perp$ and $||y|| = 1$. Now,

$$\begin{aligned}
\alpha_1 < x, Ax > +\alpha_2 < y, Ay > &= \alpha_1 < x, Ax > -\alpha_2 < x, Ax > +\alpha_2 < x, Ax > +\alpha_2 < y, Ay > \\
&= (\alpha_1 - \alpha_2) < x, Ax > +\alpha_2(< x, Ax > + < y, Ay >) \\
&= (\alpha_1 - \alpha_2) < x, Ax > +\alpha_2 tr(A) \\
&= (\alpha_1 - \alpha_2) < x, Ax > +\alpha_2 tr(A) < x, x > \\
&= < x, ((\alpha_1 - \alpha_2)A + \alpha_2 tr(A)I_2)x >
\end{aligned}$$

Thus, $W_\alpha(A) = W((\alpha_1 - \alpha_2)A + \alpha_2 tr(A)I_2)$, which is convex.

■

Lemma 5: If (α_1, α_2) is obtained by pinching (β_1, β_2), then $W_\alpha(A) \subset W_\beta(A)$.
Proof: $\beta < \alpha \implies \exists c \in [0, 1] : \alpha_1 = c\beta_1 + (1 - c)\beta_2$ and $\alpha_2 = (1 - c)\beta_1 + c\beta_2$.
Let B = $A - \frac{1}{2}tr(A)I_2$. Now, $z\ inW_\alpha(A) \implies$ that there exists a unit vector $x \in \mathbb{C}^2$ such that $z =< x, (\alpha_1 - \alpha_2)A+$

$\alpha_2 tr(A)I_2)x>$. Then,

$$
\begin{aligned}
z &= <x, ((\alpha_1-\alpha_2)A+\alpha_2 tr(A)I_2)x> \\
&= (\alpha_1-\alpha_2)<x,Ax> + \alpha_2 tr(A) \\
&= (\alpha_1-\alpha_2)<x,Ax> + \frac{1}{2}2\alpha_2 tr(A) \\
&= (\alpha_1-\alpha_2)<x,Ax> + -(\frac{1}{2}(\alpha_1-\alpha_2)tr(A)<x,x>)) + \frac{1}{2}(\alpha_1+\alpha_2)tr(A) \\
&= (\alpha_1-\alpha_2)<x,(A-\frac{1}{2}tr(A)I_2)x> + \frac{1}{2}(\alpha_1+\alpha_2)tr(A) \\
&= (\alpha_1-\alpha_2)<x,Bx> + \frac{1}{2}(\alpha_1+\alpha_2)tr(A)
\end{aligned}
$$

Thus, $W_\alpha(A) = (\alpha_1-\alpha_2)W(B) + \frac{1}{2}(\alpha_1+\alpha_2)tr(A) = (2c-1)(\beta_1-\beta_2)W(B) + \frac{1}{2}(\beta_1+\beta_2)tr(A)$.
Now, tr(B)=0. Therefore, the eigenvalues of B are {b,-b} for some b $\in \mathbb{C}$. Thus, W(B) is symmetric about the origin. And since c $\in [0,1]$, $-1 \leq (2c-1) \leq 1 \implies (2c-1)W(B) \subset W(B)$. Hence, $W_\alpha(A) = (2c-1)(\beta_1-\beta_2)W(B) + (\beta_1+\beta_2)tr(A) \subset (\beta_1-\beta_2)W(B) + \frac{1}{2}(\beta_1+\beta_2)tr(A) = W_\beta(A)$.

■

Lemma 5 can be generalized to two vectors $\alpha, \beta \in \mathbb{C}^n$, when α is obtained by pinching β, i.e, $\exists i, j and c \in [0,1]$: $\alpha_i = c\beta_i + (1-c)\beta_j, \alpha_j = (1-c)\beta_i + c\beta_j$ and $\alpha_k = \beta_k \forall k \neq i,j$.

Lemma 6: If α is obtained from β by pinching, then $W_\alpha(A) \subset W_\beta(A)$.If α is obtained from β by a finite number of pinching, then $W_\alpha(A) \subset W_\beta(A)$.
Proof: Let $\{x_1,...,x_n\}$ be an orthonormal basis of $\mathbb{C}^n$, i,j $\in \{1,...,n\}$ and V = span$\{x_i,x_j\}$. Let P:$\mathbb{C}^n \to V$ be the projection map onto V.Therefore,

$W_\alpha(A) = \{\sum_{k\neq i,j} \alpha_k < y_k, Ay_k > + W_{(\alpha_i,\alpha_j)}(PA) | \{y_1,...,y_n\}$ is an orthonormal basis of $\mathbb{C}^n$, P- projection map onto span$\{y_i,y_j\}\}$

$W_\beta(A) = \{\sum_{k\neq i,j} \alpha_k < y_k, Ay_k > + W_{(\beta_i,\beta_j)}(PA) | \{y_1,...,y_n\}$ is an orthonormal basis of $\mathbb{C}^n$, P- projection map onto span$\{y_i,y_j\}\}$

Now, by lemma 5, we get $W_{(\alpha_i,\alpha_j)}(PA) \subset W_{(\beta_i,\beta_j)}(PA)$. Thus, $W_\alpha(A) \subset W_\beta(A)$. If α was obtained from β by finite number of pinching, it is enough to repeat the above process repeatedly for each pinching.

■

Lemma 7: Let $\alpha, \beta \in \mathbb{R}^n$ with β ordered $\beta_1 \geq ... \geq \beta_n$. If α is obtained from β by pinching a finite number of times, then there exists a doubly stochastic matrix S: $\alpha = S\beta$.

Proof: In the case of single pinching, S=$[s_{k,l}]$ where $s_{k,l} = \begin{cases} 1; k \neq i \ or \ l \neq j, k=j \\ c; k=l=i \ or \ k=l=j \\ (1-c); (k=i,l=j) \ or \ (k=j,l=i) \\ 0; o/w \end{cases}$ For finite number of times, the matrix S is obtained by taking the composition of the doubly stochastic matrix involved in each pinching in order. (Doubly Stochastic matrices are closed under matrix multiplication).

■

Back to the proof, given A$\in M_n(\mathbb{C})$and C$\in \mathbb{H}_n(\mathbb{C})$. Since C is hermitian, $\exists c \in \mathbb{R}^n : C$ is unitarily similar to $diag(c) = [c]$.Therefore,
$W_C(A) = \{tr(AU^*[c]U) : U \in \mathbb{U}(n)\}$. Let M(C)=$\{U^*[c]U : U \in \mathbb{U}(n)\}$.Then, co($W_C(A)$)=$\{tr(Ax) : x \in co(M(C))\}$.
Now, $b < c \implies W_b(A) \subset W_c(A)$.

Claim: co(M(C))=$\{U^*[b]U : U \in \mathbb{U}(n), b < c\}$ (=K).
Now, M(C) $\subset$ K (since c=I_nc, i.e c<c). Let b¡c and U $\in \mathbb{U}(n)$. Then, $\exists$ doubly stochastic matrix S: b=Sc.
$\implies \exists$ permutation matrices $P_1, ..., P_k$ and $\theta_1, ..., \theta_k \in \mathbb{R}^+ : \sum_{i=1}^k \theta_i = 1$ and $S = \sum_{i=1}^k \theta_i P_i$.
$\implies b = \sum_{i=1}^k \theta_i P_i c$
$\implies [b] = \sum_{i=1}^k \theta_i P_i [c] P_i^T$
$\implies U^*[b]U = \sum_{i=1}^k \theta_i U^* P_i [c] P_i^T U = \sum_{i=1}^k \theta_i (P_i^T U)^* [c] (P_i^T U) \in co(M(C))$

Now, it is enough to show that K is convex. Let a,b<c, $U_1, U_2 \in \mathbb{U}(n)$. Let $\alpha \in [0, 1]$. Then, $\alpha U_1^*[a]U_1 + (1-\alpha)U_2^*[b]U_2$ is hermitian. Thus, there exists [d], $U \in \mathbb{U}(n)$: $U^*[d]U = \alpha U_1^*[a]U_1 + (1-\alpha)U_2^*[b]U_2$. $\implies [d] = \alpha U U_1^*[a]U_1 U^* + (1-\alpha) U U_2^*[b]U_2 U^*$.

T.S.T: diag($V^*[a]V$) $< a$ for all V $\in \mathbb{U}(n)$. Let V $\in \mathbb{U}(n)$:

$$V = \begin{bmatrix} v_1 \\ . \\ . \\ . \\ v_n \end{bmatrix}$$

Then,

$$V^*[a]V = \begin{bmatrix} v_1^* | ... | v_n^* \end{bmatrix} \begin{bmatrix} a_1 v_1 \\ . \\ . \\ . \\ a_n v_n \end{bmatrix}$$

$$diag(V^*[a]V) = \begin{bmatrix} \sum_{i=1}^n a_i |v_{i,1}|^2 \\ \sum_{i=1}^n a_i |v_{i,2}|^2 \\ . \\ . \\ \sum_{i=1}^n a_i |v_{i,n}|^2 \end{bmatrix} = \begin{bmatrix} |v_{1,1}|^2 & |v_{1,2}|^2 & ... & |v_{1,n}|^2 \\ |v_{2,1}|^2 & |v_{2,2}|^2 & ... & |v_{2,n}|^2 \\ . & . & . & . \\ . & . & . & . \\ . & . & . & . \\ |v_{n,1}|^2 & |v_{n,2}|^2 & ... & |v_{n,n}|^2 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ . \\ . \\ . \\ a_n \end{bmatrix} = S[a]$$

where S is doubly stochastic. Thus, $(U_1 U^*)^*[a](U_1 U^*) < a$ and $(U_2 U^*)^*[b](U_2 U^*) < b$.
$\implies \alpha(U_1 U^*)^*[a](U_1 U^*) + (1-\alpha)(U_2 U^*)^*[b](U_2 U^*) < \alpha a + (1-\alpha) b < c$.
Therefore, $d < c$. Hence, K is convex.

■

In conclusion, we know that when n=2, $W_c(A)$ is convex for any $c \in \mathbb{C}^2$ and for $n > 2$, $W_c(A)$ is convex for all $c \in \mathbb{R}^n$.

Chapter 4

Joint Numerical Range

4.1 Joint Numerical Range of an Operator

Let A = (A_1,...,A_m) $\in$ ($M_n\mathbb{C}$)^m .Then the Joint Numerical Range of A is described to be the set W(A), where

$$W(A) := \{(x^*A_1x, ..., x^*A_mx) : ||x|| = 1 \; and \; x \in \mathbb{C}^n\}$$

4.1.1 Properties of Joint Numerical Range

For all $A_1, ..., A_m, U \in M_n\mathbb{C}$, such that U is unitary, F= $\{A_1, ..., A_m\}$ and B =$\{B_1, ..., B_k\}$ is a basis of span(F) and for all i$\in \{1, ..., m\}$, A_i=H_i+iG_i where H_i and G_i are Hermitian matrices, then the following are true:

1. W(A_1,...,A_m) = W(U^*A_1U,...,U^*A_mU)
2. W(A_1,...,A_m) = W(A_1^T,...,A_m^T)
3. W(A_1,...,A_m) is convex iff W(B_1,...,B_k) is convex.
4. If W(A_1,...,A_m) is convex then so is W(C_1,...,C_s) for any family C = {C_1,...,C_s}$\subset$ Span(F).
5. The family F is commuting iff the basis B is commuting.
6. W(A_1,...,A_m) $\subset \mathbb{C}^m$ can be identified with W(H_1,G_1...,H_m,G_m) $\subset \mathbb{R}^{2m}$
7. For n = 2, and $H_1, ..., H_m \in \mathbb{H}_2$, W($H_1$,...,$H_m$) is convex iff Span{$I_2$,$H_1$,...,$H_m$} $\neq \mathbb{H}_2$.
8. If n $\geq$ 3 and $H_1, ..., H_m \in \mathbb{H}_n(\mathbb{C})$, then if dim(Span{I_n,H_1,...,H_m}) $\leq$ 4, then W(H_1,...,H_m) is convex.

Proof:
The proofs of (1.), (2.), (5.) and (6.) are direct.
(3.): Suppose W$(A_1, ..., A_m)$ is convex. Since $B_1, ..., B_k \in span(F)$, $\exists \alpha_{i,j} \in \mathbb{C}$ where $1 \leq i \leq n, 1 \leq j \leq k$ and$B_j = \sum_{i=1}^n \alpha_{i,j}A_i$.
Thus, $W(B_1, ..., B_k) = \{(< x, B_jx >)_j : ||x|| = 1\} = \{(< x, [\sum_{i=1}^n \alpha_{i,j}A_i]x >)_j : ||x|| = 1\}$.
Let $\mu \in (0, 1)$ and a,b $\in W(B_1, ..., B_k)$.
Then $\mu a + (1-\mu)b = \mu(< x, [\sum_{i=1}^n \alpha_{i,j}A_i]x >)_j + (1-\mu)(< y, [\sum_{i=1}^n \alpha_{i,j}A_i]y >)_j = (\mu < x, [\sum_{i=1}^n \alpha_{i,j}A_i]x > +(1-\mu) < y, [\sum_{i=1}^n \alpha_{i,j}A_i]y >)_j = \sum_{i=1}^n \alpha_{i,j}(\mu < x, A_ix > +(1-\mu) < y, A_iy >)_j$.
Since $W(A_1, ..., A_m)$ is convex, there exists z: $||z|| = 1$ and $(\mu < x, A_ix >)_i + ((1-\mu) < y, A_iy >)_i = (< z, A_iz >)_i$
Therefore $\mu a + (1-\mu)b = \sum_{i=1}^n(\alpha_{i,j}(\mu < x, A_ix > +(1-\mu) < y, A_iy >)) = \sum_{i=1}^n(\alpha_{i,j}(< z, A_iz >))_j = (< z, \sum_{i=1}^n \alpha_{i,j}A_iz >))_j = (< z, B_jz >)_j$. Thus, $W(B_1, ..., B_k)$ is convex. The other way implication proceeds in the

same way.

■

(4.): Corollary of (3.). Direct from the proof.
(7.): Let n=2 and $H_1, ..., H_m \in \mathbb{H}_2$. Now, dim($\mathbb{H}_2$)=4. Now, let B=$\{B_1, ..., B_k\}$ be a basis of $span\{H_1, ..., H_m\}$.It is enough to observe whether $W(B_1, ..., B_k)$ is convex or not. If k=1, then W(B_1) is convex by Toeplitz Hausdorff. If k=2, then again by (6.) and Toeplitz Hausdorff, W(B_1, B_2) is convex. If k=3, then either $I_2 \in span\{B_1, B_2, B_3\}$ or $I_2 \notin span\{B_1, B_2, B_3\}$. Suppose $I_2 \in span\{B_1, B_2, B_3\}$. Then, WLOG let $B_3 = I_2$. Thus, $W(B_1, B_2, I_2) = \{(< x, B_1x >, < x, b_2x >, 1) : ||x|| = 1\}$. which can be identified with $W(B_1, B_2)$ which is convex. Now for the case that $I_2 \notin span\{B_1, B_2, B_3\}$, this means $\{I_2, B_1, B_2, B_3\}$ is linearly independent. Thus, it can be represented by $I_2, \begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix} (= C_1), \begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix} (= C_2), \begin{bmatrix}0 & i\\ -i & 0\end{bmatrix} (= C_3)$. Now, it is enough to show that $W(C_1, C_2, C_3)$ is not convex.
Let $z = (z_1, z_2)\mathbb{C}^2 : ||z|| = 1$. Then, $z^*C_1z = |z_1|^2 - |z_2|^2$, $z^*C_2z = 2Re(\overline{z_1}z_2)$ and $z^*C_3z = 2Im(\overline{z_1}z_2)$.
$\implies ||(< z, C_1z >, < z, C_2z >, < z, C_3z >)||^2 = 4(Re(\overline{z_1}z_2))^2 + 4(Im(\overline{z_1}z_2))^2 + (|z_1|^2 - |z_2|^2)^2$
$= 4|(\overline{z_1}z_2)|^2 + |z_1|^4 + |z_2|^2 - 2|z_1z_2|^2 = |z_1|^4 + |z_2|^2 + 2|z_1z_2|^2 = (|z_1|^2 + |z_2|^2)^2 = 1$.
Thus, $W(C_1, C_2, C_3) \subset S^2$.
Now, $S^2 = \{(cos\phi, cos\theta sin\phi, sin\theta sin\phi) : \theta, \phi \in \mathbb{R}\}$. Let $z_1 = \sqrt{\frac{(1+cos\phi)}{2}}$, $z_2 = \sqrt{\frac{(1-cos\phi)}{2}}e^{i\theta}$. Then, $||z||^2 = \frac{(1+cos\phi)}{2} + \frac{(1-cos\phi)}{2}|e^{i\theta}| = 1$. And $|z_1|^2 - |Z_2|^2 = \frac{(1+cos\phi)}{2} - \frac{(1-cos\phi)}{2} = cos\phi$, $\overline{z_1}z_2 = \sqrt{\frac{(1-cos^2\phi)}{4}}e^{i\theta} = \frac{sin\phi}{2}(cos\theta + i\sin\theta)$.
Thus, $2Re(\overline{z_1}z_2) = cos\theta sin\phi$ and $2Im(\overline{z_1}z_2) = sin\theta sin\phi$. Thus, $W(C_1, C_2, C_3) = S^2$, $\implies W(C-1, C-2, C_3)$ is not convex.

■

(8.): It is the direct result of Theorem 1 from Result 07.

4.1.2 Result on Joint Numerical Range

Let $B_1, ..., B_r \in \mathbb{H}_n(\mathbb{C})$ be Hermitian Matrices. For every unit vector $v = (v_1, ..., v_r) \in \mathbb{R}^r$, let $P_v = \{b \in \mathbb{R}^r : b^*v \leq \lambda_1(v_1B_1 + ... + v_rB_r)\}$, where $\lambda_1(H)$ denotes the largest eigenvalue of $H \in \mathbb{H}_n(\mathbb{C})$ and $b^*v = \Sigma_{i=1}^r b_iv_i$ for $b = (b_1, ..., b_r) \in \mathbb{R}^r$. Then, Conv$W(B_1, ..., B_r) = \bigcap\{P_v : v = (v_1, ..., v_r) \in \mathbb{R}^r, ||v|| = 1\}$. Consequently, $\partial P_v \cap W(B_1, ..., B_r) = \{(x^*B_1x, ..., x^*B_rx) : x \in \mathbb{C}^n, ||x|| = 1, B_vx = \lambda_1(B_v)x\}$ where $B_v = \Sigma_{i=1}^r v_iB_i$.

Proof: Let $x \in \mathbb{C}^n$ be a unit vector and b = $(x^*B_1x, ..., x^*B_rx) \in W(B_1, ..., B_r)$. Then $\forall v \in \mathbb{R}^r$ such that v is a unit vector, $< b, v >= x^*(\sum_{i=1}^r v_iB_i)x =< x, \sum_{i=1}^r v_iB_ix >\leq \lambda_1(\sum_{i=1}^r v_iB_i)$.
Therefore, $W(B_1, ..., B_r) \subset P_v$. Now, P_v is convex since $\delta \in (0, 1)$ b,c $\in P_v \implies < \delta b + (1-\delta)c, v >= \delta < b, v > +(1-\delta) < c, v >\leq \delta\lambda_1(\sum_{i=1}^r v_iB_i) + (1-\delta)\lambda_1(\sum_{i=1}^r v_iB_i) = \lambda_1(\sum_{i=1}^r v_iB_i)$.
Thus, conv W($B_1, ..., B_r$) $\subset P_v$.
Suppose b = $(b_1, ..., b_r) \notin convW(B_1, ..., B_r)$. Then by the hyperplane separation theorem, there exists a unit vector v= $(v_1, ..., v_r)$$\in \mathbb{R}^r$ such that $\sum_{i=1}^r b_jv_j > \sum_{i=1}^r y_jv_j$ for all $y = (y_1, ..., y_r) \in W(B_1, ..., B_r)$. Now, $y \in W(B_1, ..., B_r) \implies \exists x \in \mathbb{C}^n : y_j =< x, B_jx >$. Therefore, $\sum_{i=1}^r b_iv_i > \sum_{i=1}^r v_i < x, B_ix >=< x, \sum_{i=1}^r v_jB_ix >$. I.e $< b, v >> \lambda_1(\sum_{i=1}^r v_iB_i)$. Thus, b $\notin P_v$.
Hence, conv(W($B_1, ..., B_r$)) $= \bigcap_{||v||=1} P_v$.
Now, $\partial P_v = \{b \in \mathbb{R}^r :< b, v >= \lambda_1(\sum_{i=1}^r v_iB_i)\}$. Therefore, b $\in \partial P_v \cap W(B_1, ..., B_r)$
$\implies b^*v = (< x, B_1x >, ..., < x, B_rx >)*(v_1, ..., v_r) = \sum_{j=1}^r v_j < x, B_jx >=< x, (\sum_{j=1}^r v_jB_j)x >= \lambda_1(\sum_{j=1}^r v_jB_j) \implies$ x is an eigenvector of $\sum_{j=1}^r v_jB_j$ corresponding to $\lambda_1(\sum_{j=1}^r v_jB_j)$. Thus, $\partial P_v \cap W(B_1, ..., B_r) = \{(< x, B_1x >, ..., < x, B_rx >) : x \in \mathbb{C}^n, ||x|| = 1, B_v(x) = \lambda_1(B_v)x\}$

■

RESULT 09:

For all commuting family F = $\{A_1, ..., A_m\} \subset M_2(\mathbb{C})$, W$(A_1, ..., A_m)$ is convex.

Proof:

If F contains only scalars, we are through. Suppose $\exists X \in F$ s.t. X is not a scalar. Now, WLOG let all members of F be upper triangular (By simultaneously upper triangularizing the matrices unitarily). Consider $X' = X - tr(X)I_2 = \begin{bmatrix} x_1 & x_2 \\ 0 & -x_1 \end{bmatrix}$. Then for all $Y \in F$, $Y' = Y - tr(Y)I_2 = \begin{bmatrix} y_1 & y_2 \\ 0 & -y_1 \end{bmatrix}$ commutes with X'.

Thus X'Y'=$\begin{bmatrix} x_1 & x_2 \\ 0 & -x_1 \end{bmatrix} \begin{bmatrix} y_1 & y_2 \\ 0 & -y_1 \end{bmatrix} = \begin{bmatrix} x_1y_1 & x_1y_2 - x_2y_1 \\ 0 & x_1y_1 \end{bmatrix} = \begin{bmatrix} x_1y_1 & x_2y_1 - x_1y_2 \\ 0 & x_1y_1 \end{bmatrix}$=Y'X'.

$\implies x_1y_2 = x_2y_1$. Since X is non scalar, either $x_1 \neq 0$ *or* $x_2 \neq 0$.

Suppose $x_1 \neq 0$. Then, $y_2 = \frac{x_2y_1}{x_1} \implies$ Y'=$\begin{bmatrix} y_1 & \frac{x_2y_1}{x_1} \\ 0 & -y_1 \end{bmatrix} = \frac{y_1}{x_1} \begin{bmatrix} x_1 & x_2 \\ 0 & -x_1 \end{bmatrix} = \frac{y_1}{x_1}$X'.

If $x_2 \neq 0$, then $y_1 = \frac{x_1y_2}{x_2} \implies$ Y' = $\begin{bmatrix} \frac{x_1y_2}{x_2} & y_2 \\ 0 & -\frac{x_1y_2}{x_2} \end{bmatrix} = \frac{y_2}{x_2} \begin{bmatrix} x_1 & x_2 \\ 0 & -x_1 \end{bmatrix} = \frac{y_2}{x_2}$X'.

■

4.2 Joint r-Numerical Range of an Operator

Let $1 \leq r \leq n-1$. The r-Joint Numerical Range of $B = (B_1, ..., B_m)$ is described to be the set W$^{(r)}$(B), where

$$W^{(r)}(B) := \{(\sum_{i=1}^{r} x_i^* B_1 x_i, ..., \sum_{i=1}^{r} x_i^* B_m x_i) : \ x_i \in \mathbb{C}^n \ and \ \sum_{i=1}^{r} x_i^* x_i = 1\}$$

4.2.1 Result on convexity of Joint r-Numerical Range

Let $f_{\mathbb{F}}(n)$ be the dimension of the real linear space $\mathbb{H}_n(\mathbb{F})$. Then the following theorems are equivalent.

Theorem 1:
Let $1 \leq r \leq n-1$. If p< $f_{\mathbb{F}}(r+1) - \delta_{n,r+1}$, then for any $A_1, ..., A_p \in \mathbb{H}_n(\mathbb{F})$, $W^{(r)}(A_1, ..., A_p)$ is convex where $\delta_{i,j}$ is the Kronecker delta.

Theorem 2:(Bohnenblust)
Let $1 \leq r \leq n-1$. If p< $f_{\mathbb{F}}(r+1) - \delta_{n,r+1}$, then for any $A_1, ..., A_p \in \mathbb{H}_n(\mathbb{F})$ such that $(\sum_{i=1}^{r} x_i^* A_1 x_i, ..., \sum_{i=1}^{r} x_i^* A_p x_i) \neq (0, ..., 0)$ for all $x \in (\mathbb{F}^n)^p \backslash \{(0, 0, ..., 0)\}$, then there exists $a_1, ..., a_p \in \mathbb{R}$ such that $\Sigma_{i=1}^{p} a_i A_i > 0$.

Proof of equivalence:

Let $1 \leq r \leq n-1$. Let p<f$_{\mathbb{F}}(r+1) - \delta_{n,r+1}$, $A_1, ..., A_p \in \mathbb{H}_n(\mathbb{F})$.
Suppose theorem 1 is true and let $(\sum_{i=1}^{r} x_i^* A_1 x_i, ..., \sum_{i=1}^{r} x_i^* A_p x_i) \neq (0, ..., 0)$ for all $x \in (\mathbb{F}^n)^p \backslash \{(0, 0, ..., 0)\}$.
Let A = $W^{(r)}(A_1, ..., A_p)$ and B= {(0,...,0)}. Then, by theorem 1, A is convex and B is a closed set.

Hyperplane separation theorem:

Let C and D be two disjoint nonempty convex subsets of $\mathbb{R}^n$. Then there exist a nonzero vector v and a real number c such that $< x, v> \geq c$ and $<y, v> \leq c$ for all x $\in$ C and y $\in$ D. Furthermore, if both C and D are closed, then the inequality can be taken strictly.

Now, A and B are closed and convex. Thus there exists c$\in \mathbb{R}^p, \beta \in \mathbb{R} : \forall x \in W^{(r)}(A_1, ..., A_p) : x = (\sum_{i=1}^r x_i^* A_1 x_i, ..., \sum_{i=1}^r x_i^* A_p x_i)$
$< x, c >> \beta$ and $< c, (0, 0, ..., 0) > (= 0) < \beta$. Thus, if $c = (\alpha_1, ..., \alpha_p)$, then $< x, c >= \alpha_1 \sum_{i=1}^r x_i^* A_1 x_i + ... + \alpha_p \sum_{i=1}^r x_i^* A_p x_i = \sum_{i=1}^r \sum_{j=1}^p x_i^* (\alpha_j A_j) x_i = \sum_{i=1}^r x_i^* (\sum_{j=1}^p (\alpha_j A_j)) x_i > 0$.
$\implies y^*(\sum_{j=1}^p (\alpha_j A_j)) y > 0$ for all $y : ||y|| = 1$. Thus $(\sum_{j=1}^p (\alpha_j A_j))$ is positive definite.
Theorem 1 $\implies$ Theorem 2.

Now suppose theorem 2 is true. Let D = $\{(x_1, ..., x_r) : x_i \in \mathbb{F}^n and \sum_{i=1}^r x_i^* x_i = 1\}$. Let g: $D \to \mathbb{R}^p$ given by g$((x_1, ..., x_r)) = (\sum_{i=1}^r x_i^* A_1 x_i, ..., \sum_{i=1}^r x_i^* A_p x_i)$. Thus, g(D)=$W^{(r)}(A_1, ..., A_p)$.
Let, if possible, g(D) not be convex.
Then, there exists x = $(x_1, ..., x_r)$, y = $(y_1, ..., y_r) \in D$ and $\delta \in (0, 1) : g(x) = (a_1, ..., a_p)$, $g(y) = (b_1, ..., b_p)$ and $\delta g(x) + (1 - \delta) g(y) \notin g(D)$. Let $B_j = A_j - [\delta a_j + (1 - \delta) b_j] I_n$ for $1 \leq \jmath \leq p$. Then $B_j \in \mathbb{H}_n(\mathbb{F})$ with $(\sum_{i=1}^r z_i^* B_1 z_i, ..., \sum_{i=1}^r z_i^* B_p z_i) \neq (0, 0, ..., 0)$ for all $(z_1, ..., z_r) \neq (0, 0, ..., 0)$. Therefore, by theorem 2, there exists $\beta_1, ..., \beta_p \in \mathbb{R}$: B = $\sum_{j=1}^p \beta_j B_j > 0$.
Now, $(\sum_{i=1}^r x_i^* B_1 x_i, ..., \sum_{i=1}^r x_i^* B_p x_i) = (a_1 - \delta a_1 - (1 - \delta) b_1, ..., a_p - \delta a_p - (1 - \delta) b_p) = (1 - \delta)(a_1 - b_1, ..., a_p - b_p)$ and $(\sum_{i=1}^r y_i^* B_1 y_i, ..., \sum_{i=1}^r y_i^* B_p y_i) = (b_1 - \delta a_1 - (1 - \delta) b_1, ..., b_p - \delta a_p - (1 - \delta) b_p) = (-\delta)(a_1 - b_1, ..., a_p - b_p)$.
Taking inner product with $(\beta_1, ..., \beta_p)$, we get,
$\sum_{i=1}^r x_i^* B x_i = (1 - \delta) \sum_{j=1}^p \beta_j (a_j - b_j) > 0$ and
$\sum_{i=1}^r y_i^* B y_i = (-\delta) \sum_{j=1}^p \beta_j (a_j - b_j) > 0$
Since $\delta \in (0, 1)$, either $-\delta \sum_{j=1}^p \beta_j (a_j - b_j) > 0$ or $(1 - \delta) \sum_{j=1}^p \beta_j (a_j - b_j) > 0$, exactly one of this can be true, which gives us a contradiction. Hence, Theorem 2 $\implies$ Theorem 1.

■

4.2.2 Proof of Bohnenblust Theorem

Theory behind:

Let $n \geq 2$. $V = \mathbb{H}_n(\mathbb{F})$ with the inner product $< a, b >= tr(ab)$.
$L \leq V$ is stb Positive Definite if $\exists a \geq o : a \in L$. We write $L \in \mathbb{P}$.
$L \leq V$ is stb Jointly Definite if $\epsilon^* x e = 0 \ \forall x \in L \implies \epsilon = 0$.We write $L \in \mathbb{D}_1$.
Let f(r) = $\begin{cases} r^2; \mathbb{F} = \mathbb{C} \\ \frac{r(r+1)}{2}; \mathbb{F} = \mathbb{R} \end{cases}$
Thus, dim(V)= f(n).

$L \leq V$ is stb Jointly Definite of degree r if $\sum_{i=1}^r \epsilon_i^* x e_i = 0 \ \forall x \in L \implies \epsilon_i = 0 \ \forall i$.We write $L \in \mathbb{D}_r$.

It is easy to see that $\mathbb{D}_1 \supset \mathbb{D}_2 \supset ... \supset \mathbb{D}_r ... \supset \mathbb{P}$.

Equivalent definition of $\mathbb{D}_r$:
Since $a \in V$ is stb positive semi-definite of rank $\leq r \iff \exists \epsilon_i \in \mathbb{F}^n : a = \sum_{i=1}^r \epsilon_i \epsilon_i^*$,
L $\in \mathbb{D}^r \iff$ [($a \in V$ is positive semi-definite of rank $\leq r$ and $< x, a >= 0 \ \forall x \in L$) $\implies a = o$] $\iff L^\perp$ does not contain any PSD matrix of rank $\leq r$ except for o.

Lemma: $\mathbb{P} = \mathbb{D}_n$.

Now, we know that $\mathbb{P} \subset \mathbb{D}_n$. Let $L^\perp = K$. Suppose K has no PSD matrix except o.
Since A=$x \in V : x \geq o$ is a convex cone, by Hyperplane Separation theorem $\exists$ an hyperplane K'$\supset$ K and K'$\cap$A = {O}. Thus $dim(K') = n-1$ and K' contains no PSD matrix except o. Therefore, L'=$(K')^\perp$ is a one dimensional subset of L. Let $a \neq o \in L'$. Now, if a is not definite, then there exists $\epsilon : \epsilon^* a \epsilon = 0 \implies <a, \epsilon\epsilon^*>= 0 \implies \epsilon\epsilon^* \in K'$ which is a contradiction. Hence, $\mathbb{P} = \mathbb{D}_n$.

Thus, L $\in \mathbb{P} \iff L^\perp$ does not contain any non-zero PSD matrix.

Lemma 1: Let $2 \leq r \leq n-1$, $k \geq f(n) - f(r) + 1$, $K \leq V : dim(K) = k$ *and* $a \geq o \in K$: rank(a)=r. Then K either contains
1. An indefinite matrix of rank $\leq r$
2. A PSD matrix $\neq o$ of rank $< r$.

Proof:
WLOG, let a = $\begin{bmatrix} I_r & O \\ O & O_{n-r} \end{bmatrix}$.
This can be done because suppose a is any psd matrix of rank r.
Then, there is a unitary matrix u: $u^*au = \begin{bmatrix} \Lambda_r & O \\ O & O_{n-r} \end{bmatrix}$. where Λ_r is a diagonal matrix with positive eigenvalues.
Let d = $(\Lambda_r)^{\frac{-1}{2}} \oplus I_{n-r}$ Let s= ud. Then, $s^*Ks = \{s^*xs : x \in K\}$.
Now, $(s^*xs)^* = (du^*xud)^* = du^*x^*ud = du^*xud = s^*xs \;\; \forall x \in K$ and the definiteness/indefiniteness does not change since for all non-zero vector $\epsilon \in \mathbb{F}^n$, there exists $\omega \in \mathbb{F}^n$: $\epsilon = s(\omega)$. Thus, $\omega^*(s^*xs)\omega = \epsilon^* x \epsilon$. Thus, s^*xs and x have the same type of definiteness. Thus, WLOG, we can take K' = s^*Ks and proceed to prove this for K' instead.

Let M = $\{H_r \oplus [\alpha] \oplus O_{n-r-1} : H_r \in H_r(\mathbb{F}), \alpha \in \mathbb{R}\ \}$. Then, dim(M)= f(r)+1. $\implies dim(M \cap K) \geq 1+1 = 2$.

$$\text{Thus, } \exists b = \begin{bmatrix} \beta_1 & ... & 0 & 0 & \\ . & ... & . & . & \\ . & ... & . & . & O \\ . & ... & . & . & \\ 0 & ... & \beta_r & 0 & \\ 0 & ... & 0 & \beta & \\ & O & & & O \end{bmatrix} \in (K \cap M) : \text{a and b are linearly independent with } \beta_1 \geq \beta_2 \geq ...\beta_r \text{ and } \beta > 0$$

(obtained by taking K'=U*KU where UbU* is the matrix b we find in K). Then c=$b - \beta_1 b$ has the eigenvalues 0, $\beta_2 - \beta_1, ..., \beta_n - \beta_1, \beta$, 0,0,...,0. Thus c is either a indefinite matrix with rank $\leq r$ or a PSD matrix of rank $< r$ (in the case of $\beta_1 = \beta_n$).

Lemma 2:
Let $1 \leq r \leq n, k \geq f(n) - f(r) + 1$ and $K \leq V : dim(K) = k$. If $K_1 = \{x \in K : ||x|| = 1, rank(x) \leq r\}$, then K_1 is projectively connected.

Proof:
Let A=-A and B=-B be closed sets such that $A \cup B = K_1$ and $A \cap B = \emptyset$.
Need to prove that either $A = \emptyset$ or $B = \emptyset$.
Let E(A)=$\{e \in V : e^2 = e, rank(e) = n-r, \exists a \in A : ae = o\}$.
Let E(B)=$\{e \in V : e^2 = e, rank(e) = n-r, \exists b \in B : be = o\}$
Now, E(A) and E(B) are closed by definition and E(A)$\cup$E(B)=E(K_1).

Let e be any idempotent matrix of V of rank n-r. Then E= $\{x \in V | xe = o\}$ has dimension f(r). $\implies$ dim($K \cap E$)$\geq$ 1. Thus, E(K_1)=$\{e \in V | e^2 = e, rank(e) = n-r, \exists x \in K_1 : xe = o\}$=$\{e \in V | e^2 = e, rank(e) = n-r\}$.

Thus E(K_1) is connected. $\implies E(A) =$ or $E(B) = \emptyset$. If E(A)=, then as E(A) $\cup$ E(B) = E(K_1), A = $\emptyset$ and vice versa. Thus,K_1 is projectively connected.

Theorem: Let K be a k-dimensional subspace of V which contains a non-zero semi definite matrix. If $2 \leq r \leq n$ and $k \geq f(n) - f(r) + \delta_{n,r} + 1$, then K contains a non-zero PSD matrix of rank $< r$.

Proof:
Let a be the given psd matrix contained in K. Let rank(a)=R. Note, that for all possible values of n and r, k$\geq$ 2. Then, if R=n, we get that since k$\geq$ 2, there exists a c in K such that a and c are linearly independent. Now consider the matrix a+δc. For small enough δ, it is a positive matrix. As the set of all psd matrices form a convex cone with its interior containing all the positive matrices and having a boundary containing all psd matrices which are not strictly positive , we get that there exists $\epsilon \in F^+$: a+ ϵc is positive semi-definite with rank < n. We can instead take this psd matrix as a.
Now, if $R < r$, we are through. Thus, let $r \leq R \leq n-1$. Suppose r = n. Then a is the required psd matrix.
If $2 \leq r \leq n-1$, then let A = $\{x \in K : ||x|| = 1, rank(x) \leq R \; and \; x \; is \; a \; definite \; matrix\}$ and let B = cl($\{x \in K : ||x|| = 1, rank(x) \leq R \; and \; x \; is \; an \; indefinite \; matrix\}$). Now, if $K_1 = \{x \in K : ||x|| = 1, rank(x) \leq R\}$, then we get that A=-A, B=-B with $A \cup B = K_1$ and since $R \geq r$, $-f(r) \geq -f(R)$. I.e, $k \geq f(n) - f(r) + 1 \geq f(n) - f(R) + 1$. Now, A is non-empty since $a \in A$.
Suppose $A \cap B \neq \emptyset$. Then there exists a semi definite matrix b in K of rank $\leq R$ and such that $||b|| = 1$ which is the limit of a sequence of indefinite matrices from B .As indefinite matrices have to have both positive and negative eigenvalues, by the continuity of eigenvalues of Hermitian matrices, we get that rank(b) has to be $\leq R$. (Since at least one of the eigenvalues which is positive or negative has to disappear)
Suppose $A \cap B = \emptyset$. Then, by lemma 2, we get that $B = \emptyset$. This means that the set of all indefinite matrices of rank less that R is empty. Thus, by lemma 1, there exists a non-zero psd matrix b of rank $< R$.
Now, if rank(b)$< r$, we are through. If rank(b) $\geq r$, then take rank(b)=R and continue the above process till we get a psd matrix p of rank $< r$.

Main Theorem: If $1 \leq r \leq n-1$ and $p < f(r+1) - \delta_{n,r+1}$, then $\mathbb{P}^{(p)} = \mathbb{D}_r^{(p)}$ where $L \in \mathbb{P}^{(p)} \iff L \in \mathbb{P}$ and dim(L) = p and $L \in \mathbb{D}_r^{(p)} \iff L \in \mathbb{D}_r$ and dim(L) = p.

Proof:

We know that $\mathbb{P}^{(p)} \subset \mathbb{D}_r^{(p)}$.Let $L \in D_r^{(p)}$. Let K=$L^\perp$ and R=r+1. Then dim(K)=f(n)-p $\geq$ f(n)-f(R)+1.
Now, by the previous theorem, if suppose K contained a non-zero PSD matrix, then K contains a non-zero PSD matrix of rank $\leq r < r+1 = R$. $\implies \impliedby$. Thus, K = $L^\perp$ contains no non-zero PSD matrices. Hence, $L^\perp \in \mathbb{P}^{(p)}$.

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4.3 Prospective Work on Joint Numerical Ranges

:

1. Investigate the properties of the Joint Numerical Range of normal matrices.
2. Explore the properties of the Joint Numerical Range of unitary matrices.
3. Examine whether the results from (1) and (2) can be unified to generalize for any tuple of matrices using Singular Value Decomposition (SVD).

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